A general bulk service queue with arrival rate dependent on server breakdowns

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A single-server queueing system with general bulk service that alternates stochastically between operational state and failed (or repair) state is considered. The system, when operational, functions as a single-server Poisson queue with general bulk service. When it fails, no service occurs but customers continue to arrive according to a Poisson process; however, the arrival rate is different from that when the system is operational. The durations of the operating periods and repair periods follow exponential and phase-type distributions, respectively. The steady-state probability vector of the number of customers in the queue and the stability condition are obtained using a matrix-geometric algorithmic approach.

Keywords: general bulk service, matrix-geometric algorithmic approach, generating function, phase-type distribution

1. Introduction

The server in many queueing systems (e.g., a computer facility) is subject to breakdown. If the server is not replaced or repaired until a breakdown occurs and the breakdowns are unpredictable in nature, then the server is not able to provide uninterrupted service to its customers. It is then necessary to see how the breakdowns affect the level of performance of the system.

Avi-Itzhak and Naor[1] and Gaver[2] have studied a single-server Poisson queue with server breakdowns. The service and repair times follow a general distribution. The steady-state waiting time distribution and average queue length have been obtained. However, the arrival rate is not dependent on operational state or repair state of the server.

Yechiali and Naor[3] have considered a single-server exponential queueing model with arrival rate depending on operational state or breakdown state of the server. The steady-state mean queue length is obtained for a system with infinite capacity. Fond and Ross[4] analyzed the same model with the assumption that any arrival finding the server busy is lost, and they obtained the steady-state proportion of customers lost.

Shogan[5] has dealt with a single-server queueing model with arrival rate dependent on server breakdowns. When the system is operational, it functions as a single-server Poisson queue with Erlang K-distributed service times and when it is not, no service takes place but customers continue to arrive according to a Poisson process with the arrival rate different from that when the system is operational. The operational period and failed period follow exponential and Erlang distributions, respectively. Steady-state waiting time distribution and average queue length have been obtained using the generating function technique.

Neuts and Lucantoni[6] have studied a Markovian queueing model with N servers subject to breakdowns and repairs. Shanthikumar[7] has analyzed a single-server Poisson queue with service time following a general distribution, time- and operation-dependent server failures, and arrival rate dependent on the state of the server.

This paper analyzes an exponential single-server general bulk service queueing model with arrivals depending on server breakdowns. The steady-state probability vec-

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tor of the number of customers in the queue and the stability condition are obtained using the matrix-geometric method of Neuts.[8] Numerical results are also presented when the repair time follows generalized Erlang and hyperexponential distributions.

This model can be fitted into real-life situations. For example, consider a tourist bus in a bus stand, which is subject to breakdown. The customers arrive at the bus stand, form a queue, and are served according to the general bulk service rule. If the bus requires repair while on tour, the tour is disrupted and cannot continue until repairs are completed. During the repair time, the customers continue to arrive at the bus stand but at a lesser arrival rate.

2. The model

Consider a single-server general bulk service queueing system with the following characteristics:

1. The system alternates between two states, the operational state and the failed (or repair) state.
2. When the system is in the operational state, it functions as a single-server Poisson queue with general bulk service, i.e., the customers arrive according to a Poisson process with rate λ, and service is done according to the general bulk service rule introduced by Neuts[9] with rate μ.
3. When the system fails while providing service to a batch of customers, the service done is lost and begins again as soon as the repair period ends.
4. During the repair period, no service takes place but customers continue to arrive according to a Poisson process with a different rate λ_1 where $\lambda_1 \leq \lambda$.
5. The duration of the operating period is exponential with mean $1/\alpha$, and the duration of the repair period is phase type with m phases.

Phase-type distributions have been introduced by Neuts.[10] A probability distribution on $(0, \infty)$ is said to be of phase type, if it is obtained as the distribution of the time until absorption in a finite Markov chain with continuous parameters. Phase-type distributions are nothing but a generalization of the Erlang distribution. In the Erlang distribution, there are m sequential phases, and the repair starts as soon as the mth phase is completed. In a phase-type distribution, there are also m phases but phases may be skipped or repeated and it is not necessary to start with phase 1 and end with phase m.

If the repair period is of phase type, one has the following situation. The repair time starts immediately after a breakdown with probability β_j in phase j, $j = 1, 2, \ldots, m$. During the repair time, the repair process transits from phase i to phase j at a rate t_{ij}, $i, j = 1, 2, \ldots, m$. Moreover, while the process is in phase i, the repair time ends at a rate t_{io}, $i = 1, 2, \ldots, m$, and the system becomes operational.

We denote by I_n the identity matrix of order n, by $\underline{0}$ the zero matrix of suitable dimension, and by $\underline{e}$ a column matrix of suitable dimension with all its components equal to one. For i and j varying from 1 to m, $T = [t_{ij}]_{mxm}$, $T_o = [t_{io}]_{mx1}$ and, $\underline{\beta} = [\beta_1, \beta_2, \ldots, \beta_m]$ such that $T\underline{e} + T_o = \underline{0}$. $(\underline{\beta}, T)$ is called the representation of the phase-type distribution. The class of phase-type distributions includes a number of important distributions such as generalized Erlang and hyperexponential distributions.

The queueing model under consideration can be studied as a continuous time Markov chain with state space $\{(0, j, 0): 0 \leq j \leq a - 1\} U \{(i, j): 0 \leq i \leq m, j \geq 0\}$. The state $(0, j, 0)$ denotes that the system is in the operational state, j customers are waiting in the queue, and the server is idle. In the state (i, j), $i = 0$ denotes that the system is in operational state and the server busy, whereas $i = 1, 2, \ldots, m$ denotes that the system is at the ith repairing phase and j denotes the number of customers in the queue.

The matrix-geometric procedure requires the generation of all possible states and, subsequently, the generation of the rate matrix Q. The infinitesimal generator of the continuous time Markov chain with the above state space has the following block partitioned structure:

$$Q = \begin{array}{c c} & \begin{array}{ccccccc} \underline{0} & \underline{1} & \cdots & \underline{a-1} & \underline{a} & \underline{a+1} & \cdots \end{array} \\ \begin{array}{c} \underline{0} \\ \underline{1} \\ \vdots \\ \underline{a-2} \\ \underline{a-1} \\ \underline{a} \\ \underline{a+1} \\ \vdots \\ \underline{b} \\ \underline{b+1} \\ \vdots \\ \underline{b+a-1} \\ \underline{b+a} \\ \underline{b+a+1} \\ \vdots \end{array} & \begin{bmatrix} D_0 & C_0 & & & & & \\ & D_0 & \ddots & & & & \\ & & \ddots & & & & \\ & & & C_0 & & & \\ B_0 & & & D_0 & C_1 & & \\ B_1 & & & & A_1 & A_0 & \ddots \\ B_1 & & & & & A_1 & \ddots \\ \vdots & & & & & & \ddots \\ B_1 & & & & & & \\ & B_1 & & & & & \\ & & \ddots & & & & \\ & & & B_1 & & & \\ & & & & A_2 & & \\ & & & & & A_2 & \\ & & & & & & \ddots \end{bmatrix} \end{array} \tag{1a}$$

where $\underline{i} = (0, i, 0)U(k, i)$ for $0 \leq i \leq a - 1$, $0 \leq k \leq m$,
$= (k, i)$ for $i \geq a$, $0 \leq k \leq m$
and all the unmarked entries are zeros.

The submatrices are defined as below. The dimensionalities of A_0, A_1, and A_2 are $(m + 1) \times (m + 1)$, B_1 is $(m + 1) \times (m + 2)$, C_1 is $(m + 2) \times (m + 1)$, and B_0, C_0, and D_0 are all $(m + 2) \times (m + 2)$. More specifically,

$$A_0 = \begin{bmatrix} \lambda & \underline{0} \\ \underline{0} & \lambda_1 I_m \end{bmatrix}; A_1 = \begin{bmatrix} -\lambda - \alpha - \mu & \alpha\underline{\beta} \\ \underline{0} & T - \lambda_1 I_m \end{bmatrix};$$

$$A_2 = \begin{bmatrix} \mu & \underline{0} \\ T_0 & \underline{0} \end{bmatrix} \tag{1b}$$

$$B_0 = \begin{bmatrix} 0 & \lambda & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} \end{bmatrix}; B_1 = \begin{bmatrix} 0 & \mu & \underline{0} \\ \underline{0} & T_0 & \underline{0} \end{bmatrix};$$

$$C_0 = \begin{bmatrix} \lambda I_2 & \underline{0} \\ \underline{0} & \lambda_1 I_m \end{bmatrix}; \tag{1c}$$

$$C_1 = \begin{bmatrix} 0 & \underline{0} \\ \lambda & \underline{0} \\ \underline{0} & \lambda_1 I_m \end{bmatrix};$$

$$D_0 = \begin{bmatrix} -\lambda-\alpha & 0 & \alpha\underline{\beta} \\ \mu & -\lambda-\alpha-\mu & \alpha\underline{\beta} \\ T_0 & \underline{0} & T-\lambda_1 I_m \end{bmatrix} \tag{1d}$$

3. The steady-state probability vector and the stability condition

Let $\underline{X}$ be the vector of steady-state probabilities associated with Q, such that

$$\underline{X}Q = \underline{0} \quad \text{and} \quad \underline{X}\underline{e} = 1 \tag{2}$$

Let us partition $\underline{X}$ as $\underline{X} = (\underline{X}_0, \underline{X}_1, \underline{X}_2, \ldots)$ where $\underline{X}_i$ for $0 \le i \le a-1$ are $1 \times (m+2)$ vectors and $\underline{X}_i$ for $i \ge a$ are $1 \times (m+1)$ vectors. Following Neuts,[8] we examine the existence of a solution of the form

$$\underline{X}_i = \underline{X}_a R^{i-a}, \; i \ge a \tag{3}$$

For this, we find from (2)

$$A_0 + RA_1 + R^{b+1}A_2 = \underline{0} \tag{4}$$

The square matrix R is of order $(m+1)$ and is the minimal nonnegative solution to the matrix nonlinear equation (4), Wallace.[11]

The eigenvalue of R with largest modulus, called the spectral radius, is less than one. Following Theorem 1 of Latouche and Neuts,[12] the matrix R is computed by successive substitutions in the recurrence relation:

$$R(0) = 0$$

$$R(n+1) = -A_0A_1^{-1} - R(n)^{b+1}A_2A_1^{-1} \quad \text{for } n \ge 0 \tag{5}$$

and is the limit of the monotonically increasing sequence of matrices $\{R_n, n \ge 0\}$.

Consider the infinitesimal generator

$$A = A_0 + A_1 + A_2,$$

which is a square matrix of order $(m+1)$. A is irreducible and there is a unique row vector $\underline{\pi} = [\pi_0, \pi_1, \pi_2, \ldots, \pi_m] \ge 0$ such that

$$\underline{\pi}A = \underline{0} \quad \text{and} \quad \underline{\pi}\underline{e} = 1 \tag{6}$$

Following Neuts,[8] the system is stable if and only if

$$\underline{\pi}A_0\underline{e} < b\underline{\pi}A_2\underline{e} \tag{7}$$

i.e., the system is stable if and only if

$$\lambda\pi_0 + \lambda_1\underline{\pi}_1\underline{e} < b(\mu\pi_0 + \underline{\pi}_1 T_0) \tag{8}$$

where $\underline{\pi}_1 = [\pi_1, \pi_2, \ldots, \pi_m]$.

When the duration of the repair period follows the Erlang distribution with mean $1/\beta$ and shape parameter m, by solving (6), we obtain

$$\pi_0 = \frac{\beta}{\alpha+\beta} \quad \text{and} \quad \pi_1 = \pi_2 = \cdots = \pi_m = \frac{\alpha}{m(\alpha+\beta)} \tag{9}$$

In this case, the stability condition (7) takes the form

$$\lambda\beta + \lambda_1\alpha < b\beta(\mu+\alpha) \tag{10}$$

If there is no breakdown ($\alpha = 0$), the stability condition (10) reduces to the well-known form

$$\lambda < b\mu \tag{11}$$

Finally, we have to determine the vectors $\underline{X}_0$, $\underline{X}_1$, $\underline{X}_2, \ldots, \underline{X}_a$. We define Q^* by

$$Q^* = \begin{array}{c} \\ \underline{0} \\ \underline{1} \\ \vdots \\ \underline{a-2} \\ \underline{a-1} \\ \\ \underline{a} \end{array} \begin{array}{c} \begin{array}{cccccc} \underline{0} & \underline{1} & \cdots & \underline{a-1} & & \underline{a} \end{array} \\ \begin{bmatrix} D_0 & C_0 & & & & \\ & D_0 & \ddots & & & \\ & & \ddots & & & \\ & & & C_0 & & \\ B_0 & & & D_0 & & C_1 \\ & & & & & \\ \sum_{i=0}^{b-a} R^iB_1 & \overset{b-a+1}{RB_1} & \cdots & \overset{b-1}{RB_1} & & \overset{b}{A_1 + RA_2} \end{bmatrix} \end{array} \tag{12}$$

We use the following lemma to later solve for the $\underline{X}_i$ vectors for $0 \le i \le a-1$.

Lemma: Q^* is an infinitesimal generator.

Proof: To prove that $Q^*\underline{e} = \underline{0}$, it is sufficient to consider the last row of Q^*, as the other rows are identical to that of Q and Q is an infinitesimal generator.

(Last row sum of Q^*)$\underline{e}$

$$= \sum_{i=0}^{b-a} R^iB_1\underline{e} + R^{b-a+1}B_1\underline{e} + \cdots + R^{b-1}B_1\underline{e} + R^bA_2\underline{e} + A_1\underline{e}$$

$$= (I-R)^{-1}[(I-R^{b+1})A_2\underline{e} + (I-R)A_1\underline{e}] \tag{13}$$

Because $(A_0 + A_1 + A_2)\underline{e} = \underline{0}$ and $B_1\underline{e} = A_2\underline{e}$ from (1a)

$$= (I-R)^{-1}[-R^{b+1}A_2\underline{e} - RA_1\underline{e} - A_0\underline{e}]$$

$$= \underline{0} \text{ using equation (4).}$$

Therefore, Q^* is an infinitesimal generator and is also irreducible.

Let $\underline{X}^* = (\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_a)$ be a solution of the equation $\underline{X}^*Q^* = \underline{0}$. Then we have using the above lemma:

$$\begin{aligned} \underline{X}_0D_0 + \underline{X}_{a-1}B_0 + \underline{X}_a\sum_{i=0}^{b-a} R^iB_1 &= 0 \\ \underline{X}_0C_0 + \underline{X}_1D_0 + \underline{X}_aR^{b-a+1}B_1 &= 0 \\ \cdots\cdots \\ \cdots\cdots \\ \underline{X}_{a-2}C_0 + \underline{X}_{a-1}D_0 + \underline{X}_aR^{b-1}B_1 &= 0 \\ \underline{X}_{a-1}C_1 + \underline{X}_a(A_1 + R^bA_2) &= 0 \end{aligned} \tag{14}$$

The vectors $\underline{X}_i$ for $0 \le i \le a-1$ can be obtained in terms of $\underline{X}_a$ from the above set of equations and $\underline{X}_a$ can be obtained from the normalizing condition

$$\sum_{i=0}^{a-1} \underline{X}_i\underline{e} + \underline{X}_a(I-R)^{-1}\underline{e} = 1 \tag{15}$$

4. Numerical results

Generalized Erlang distribution for repair time

1. For the parameters $m = 2$, $\lambda = 10$, $\lambda_1 = 6$, $\mu = 3$, $\alpha = 5$, $a = 4$, $b = 8$, and $\underline{\beta} = [1, 0]$, the matrix R is obtained from equation (5) as

$$R = \begin{bmatrix} 0.5806969482 & 0.3226094311 & 0.0967828293 \\ 0.0138208401 & 0.6743449116 & 0.2023034794 \\ 0.0026622137 & 0.0014790077 & 0.6004437023 \end{bmatrix}$$

The vectors $\underline{X}_i$ for $0 \le i \le 3$ are row vectors of order 4 and $\underline{X}_i$ for $i \ge 4$ are row vectors of order 3. Now $\underline{X}^* = (\underline{X}_0, \underline{X}_1, \underline{X}_2, \underline{X}_3, \underline{X}_4)$. Using $\underline{X}^* Q^* = \underline{0}$, the system of governing equations is obtained, whose solution is given by

$$\underline{X}_0 = \begin{bmatrix} 0.0143014739 \\ 0.0559057616 \\ 0.0390040197 \\ 0.0117012059 \end{bmatrix}^T, \underline{X}_1 = \begin{bmatrix} 0.0229425371 \\ 0.0345115045 \\ 0.0579215918 \\ 0.0243972011 \end{bmatrix}^T$$

$$\underline{X}_2 = \begin{bmatrix} 0.0288952495 \\ 0.0217770861 \\ 0.0667656921 \\ 0.0346680283 \end{bmatrix}^T, \underline{X}_3 = \begin{bmatrix} 0.0332835350 \\ 0.0140442778 \\ 0.0708036907 \\ 0.0420419242 \end{bmatrix}^T$$

$$\underline{X}_4 = \begin{bmatrix} 0.0092459603 \\ 0.0523391051 \\ 0.0409268844 \end{bmatrix}^T$$

Because the matrix R and the vector $\underline{X}_4$ are known, the remaining vectors $\underline{X}_i$, $i \ge 4$ are evaluated using the relation

$$\underline{X}_i = \underline{X}_4 R^{i-4}, \; i \ge 4$$

obtained from (3).

It may be noted that $\underline{X}_k \to 0$ as $k \to \infty$. For the numerical parameters chosen above, $\underline{X}_{66} \to 0$ and the sum of the steady-state probabilities is found to be $0.9999999995 \simeq 1$.

2. For the parameters $m = 3$, $\lambda = 20$, $\lambda_1 = 15$, $\mu = 5$, $\alpha = 7$, $a = 5$, $b = 10$, $\underline{\beta} = [1, 0, 0]$, the matrix R is obtained as

$$R = \begin{bmatrix} 0.6740255718 & 0.2483252107 & 0.0496650415 & 0.0118250099 \\ 0.0511368816 & 0.8083135880 & 0.1616627187 & 0.0384911235 \\ 0.0251960154 & 0.0092827425 & 0.7518565485 & 0.1790134639 \\ 0.0063045756 & 0.0023227384 & 0.0004645477 & 0.7143963209 \end{bmatrix}$$

The vectors $\underline{X}_i$ for $0 \le i \le 4$ are row vectors of order 5, and $\underline{X}_i$ for $i \ge 5$ are row vectors of order 4. Now $\underline{X}^* = (\underline{X}_0, \underline{X}_1, \underline{X}_2, \underline{X}_3, \underline{X}_4, \underline{X}_5)$. Using $\underline{X}^* Q^* = \underline{0}$, the system of governing equations is obtained, whose solution is given by

$$\underline{X}_0 = \begin{bmatrix} 0.0042129168 \\ 0.0221938196 \\ 0.0097287976 \\ 0.0019457595 \\ 0.0004632761 \end{bmatrix}^T, \underline{X}_1 = \begin{bmatrix} 0.0064894184 \\ 0.0164552116 \\ 0.0161339144 \\ 0.0046861025 \\ 0.0014466502 \end{bmatrix}^T$$

$$\underline{X}_2 = \begin{bmatrix} 0.0077987109 \\ 0.0127507489 \\ 0.0203081545 \\ 0.0075762078 \\ 0.0028371806 \end{bmatrix}^T, \underline{X}_3 = \begin{bmatrix} 0.0086795648 \\ 0.0103036601 \\ 0.0230265732 \\ 0.0102874705 \\ 0.0044759553 \end{bmatrix}^T$$

$$\underline{X}_4 = \begin{bmatrix} 0.0094096908 \\ 0.0086344115 \\ 0.0248267008 \\ 0.0126809430 \\ 0.0062163831 \end{bmatrix}^T, \underline{X}_5 = \begin{bmatrix} 0.0074480751 \\ 0.0223440546 \\ 0.0139795182 \\ 0.0077687302 \end{bmatrix}^T$$

Because the matrix R and the vector $\underline{X}_5$ are known, the remaining vectors $\underline{X}_i$, $i \ge 5$ are evaluated from the relation

$$\underline{X}_i = \underline{X}_5 R^{i-5}, \; i \ge 5$$

obtained from (3).

It may be noted that $\underline{X}_k \to 0$ as $k \to \infty$. For the numerical parameters chosen above, $\underline{X}_{215} \to 0$ and the sum of the steady-state probabilities is found to be $0.9999999967 \simeq 1$.

Hyperexponential distribution for repair time

1. For the parameters $m = 2$, $\lambda = 15$, $\lambda_1 = 10$, $\mu = 2$, $\alpha = 4$, $a = 4$, $b = 8$, $\underline{\beta} = [0.5 \;\; 0.5]$, the matrix R is obtained as

$$R = \begin{bmatrix} 0.7861332892 & 0.1429333280 & 0.1209435852 \\ 0.0641390104 & 0.9207525476 & 0.0098675403 \\ 0.0226753943 & 0.0041227990 & 0.7727192915 \end{bmatrix}$$

The vectors $\underline{X}_i$ for $0 \le i \le 3$ are row vectors of order 4 and $\underline{X}_i$ for $i \ge 4$ are row vectors of order 3. Now $\underline{X}^* = (\underline{X}_0, \underline{X}_1, \underline{X}_2, \underline{X}_3, \underline{X}_4)$. Using $\underline{X}^* Q^* = \underline{0}$, the system of governing equations is obtained, whose solution is given by

$$\underline{X}_0 = \begin{bmatrix} 0.0021004127 \\ 0.0145861991 \\ 0.0030339294 \\ 0.0025671710 \end{bmatrix}^T, \underline{X}_1 = \begin{bmatrix} 0.0039256366 \\ 0.0120577609 \\ 0.0056641899 \\ 0.0044337312 \end{bmatrix}^T$$

$$\underline{X}_2 = \begin{bmatrix} 0.0055111103 \\ 0.0101764499 \\ 0.0080015472 \\ 0.0058240333 \end{bmatrix}^T, \underline{X}_3 = \begin{bmatrix} 0.0068939584 \\ 0.0087650185 \\ 0.0101212206 \\ 0.0068890990 \end{bmatrix}^T$$

$$\underline{X}_4 = \begin{bmatrix} 0.0076958509 \\ 0.0106003552 \\ 0.0064832829 \end{bmatrix}^T$$

Because the matrix R and the vector $\underline{X}_4$ are known, the remaining vectors $\underline{X}_i$, $i \geq 4$ are evaluated using the relation

$$\underline{X}_i = \underline{X}_4 R^{i-4}, \; i \geq 4$$

obtained from (3).

It may be noted that $\underline{X}_k \to 0$ as $k \to \infty$. For the numerical parameters chosen above, $\underline{X}_{744} \to 0$ and the sum of the steady-state probabilities is found to be $0.9999999915 \simeq 1$.

2. For the parameters $m = 3$, $\lambda = 25$, $\lambda_1 = 20$, $\mu = 3$, $\alpha = 5$, $a = 5$, $b = 10$, $\underline{\beta} = [0.4, 0.3, 0.3]$, the matrix R is obtained as

$$R = \begin{bmatrix} 0.8090090781 & 0.0735462820 & \\ 0.0548737720 & 0.9140794340 & \\ & 0.0505630689 & 0.0466736021 \\ & 0.0034296109 & 0.0031657946 \\ 0.0282833321 & 0.0025712121 & \\ 0.0149128993 & 0.0013557182 & \\ & 0.8351010416 & 0.0016317307 \\ & 0.0009320562 & 0.7700911288 \end{bmatrix}$$

The vectors $\underline{X}_i$ for $0 \leq i \leq 4$ are row vectors of order 5, and $\underline{X}_i$ for $i \geq 5$ are row vectors of order 4. Now $\underline{X}^* = (\underline{X}_0, \underline{X}_1, \underline{X}_2, \underline{X}_3, \underline{X}_4, \underline{X}_5)$. Using $\underline{X}^* Q^* = \underline{0}$, the system of governing equations is obtained, whose solution is given by

$$\underline{X}_0 = \begin{bmatrix} 0.0044112814 \\ 0.0341205776 \\ 0.0035028963 \\ 0.0024082412 \\ 0.0022229919 \end{bmatrix}^T, \quad \underline{X}_1 = \begin{bmatrix} 0.0083010169 \\ 0.0284899354 \\ 0.0065290832 \\ 0.0043063022 \\ 0.0038325487 \end{bmatrix}^T$$

$$\underline{X}_2 = \begin{bmatrix} 0.0117119509 \\ 0.0240360292 \\ 0.0091853465 \\ 0.0058228339 \\ 0.0050104978 \end{bmatrix}^T, \quad \underline{X}_3 = \begin{bmatrix} 0.0146960266 \\ 0.0204907489 \\ 0.0115491128 \\ 0.0070515351 \\ 0.0058842354 \end{bmatrix}^T$$

$$\underline{X}_4 = \begin{bmatrix} 0.0173068162 \\ 0.0176491754 \\ 0.0136770109 \\ 0.0080610287 \\ 0.0065430267 \end{bmatrix}^T, \quad \underline{X}_5 = \begin{bmatrix} 0.0153544205 \\ 0.0138295027 \\ 0.0076771752 \\ 0.0059189284 \end{bmatrix}^T$$

Because the matrix R and the vector $\underline{X}_5$ are known, the remaining vectors $\underline{X}_i$, $i \geq 5$ are evaluated using the relation

$$\underline{X}_i = \underline{X}_5 R^{i-5}, \; i \geq 5$$

obtained from (3).

It may be noted that $\underline{X}_k \to 0$ as $k \to \infty$. For the numerical parameters chosen above, $\underline{X}_{364} \to 0$ and the sum of the steady-state probabilities is found to be $0.9999999979 \simeq 1$.

Following corollary 1 of Latouche and Neuts,[12] the calculation of the matrix R is checked in all the four cases discussed above, using $A_0 \underline{e} = (\sum_{i=1}^{b} R^i) A_2 \underline{e}$.

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